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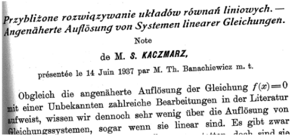
Randomized sparse Kaczmarz methods

Dirk Lorenz, joint with Frank Schöpfer, Feb 9, 2018

Inverse Problems and Machine Learning, Caltech 2018

- **The Kaczmarz method**
- Randomization
- Sparsity
- Split feasibility problems
- Convergence rates

Just solving systems of linear equations


Przybliżone rozwiązywanie układów równań liniowych. — Angenäherte Auflösung von Systemen linearer Gleichungen.

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de M. S. KACZMARZ,

présentée le 14 Juin 1937 par M. Th. Banachiewicz m. t.

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- $Ax = b$ pretty arbitrary (but consistent), m rows, n columns

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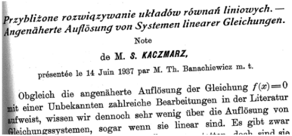
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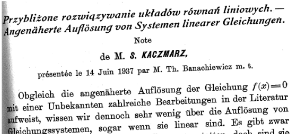
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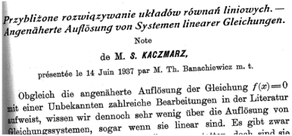
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- Stefan Kaczmarz [1937]: Convergent to some solution for all consistent systems

Learning with Kaczmarz

- Unknown distibution ρ on $X \times Y = \mathbf{R}^d \times \mathbf{R}$, regression function $f_\rho(a) = \int y \, d\rho(y \mid a)$
- Hypothesis space $\mathcal{H} = \{f_x \in L^2_{\rho_X}, x \in \mathbf{R}^d\}, f_x(a) = \langle a, x\rangle$
- Learning: Obtain samples $a \in X'$, $b \in Y$ sequentially and try to learn x
- Kaczmarz: Update x^k by

$$x^{k+1} = x^k - \frac{\langle x^k, a\rangle - b}{\|a\|^2} a$$

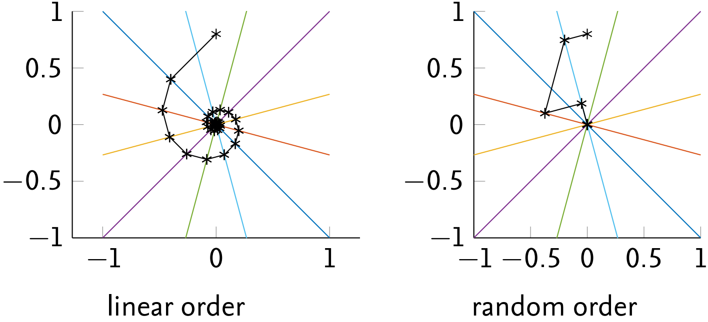
- Goal: Show that x^k converges to some x^* such that

$$f_{x^*} = \operatorname*{argmin}_{f \in \mathcal{H}} \mathbf{E}(f) = \operatorname*{argmin}_{f \in \mathcal{H}} \int_{X \times Y} (b - f(a))^2 d\rho$$

  [Lin, Zhou 2015]
- Here focus on Kaczmarz as an algorithm for solving systems

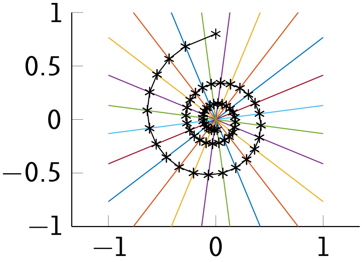
Convergence speed?

$m = 6$ rows, $n = 2$ columns:

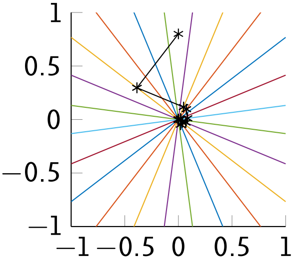


Convergence speed?

$m = 12$ rows, $n = 2$ columns:



linear order



random order

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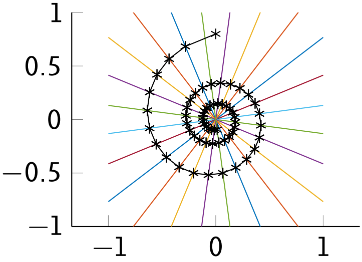

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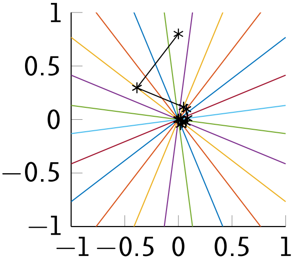

random order

- Btw: Randomized Kaczmarz is stochastic gradient descent for $\sum_i (\langle a_i, x\rangle - b_i)^2$

- The Kaczmarz method
- **Randomization**
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Randomization leads to linear convergence

- In each iteration, choose index i with probability p_i.
- If $\hat{x}$ solves (i.e. $\langle \hat{x}, a_i\rangle = b_i$), then

$$\|x^{k+1} - \hat{x}\|^2 = \|x^k - \hat{x}\|^2 - \frac{(\langle x^k - \hat{x}, a_i\rangle)^2}{\|a_i\|^2}.$$

- Taking the expectation over the choice of i gives

$$\begin{aligned}\mathbf{E}(\|x^{k+1} - \hat{x}\|^2) &= \|x^k - \hat{x}\|^2 - \sum_i p_i \frac{(\langle x^k - \hat{x}, a_i\rangle)^2}{\|a_i\|^2}\\ &= \|x^k - \hat{x}\|^2 - \langle A(x^k - \hat{x}), DA(x^k - \hat{x})\rangle\end{aligned}$$

  with $D = \operatorname{diag}(p_i/\|a_i\|^2)$.
- Gives uniform improvement

$$\mathbf{E}(\|x^{k+1} - \hat{x}\|^2) \le (1-\lambda)\|x^k - \hat{x}\|^2, \quad \lambda = \lambda_{\min}(A^T DA)$$

Theorem

$A \in \mathbf{R}^{m\times n}$, $m \geq n$ with full rank, $A\hat{x} = b$, then iterates of randomized Kaczmarz fulfill

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- Experimentally: above p not optimal, other p give larger λ

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- Convergence to minimum-norm solution $\hat{x}$

- The Kaczmarz method
- Randomization
- **Sparsity**
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Kaczmarz converging to sparse solutions?

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- Kaczmarz

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- Sparse Kaczmarz

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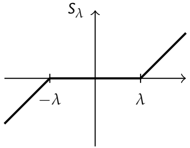
$$x^{k+1} = S_\lambda(z^{k+1})$$



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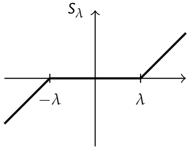

- **Theorem** [L, Schöpfer, Wenger, Magnor 2014]: The sequence x^k, when initialized with $x^0 = 0$, converges to the solution of

$$\min \|x\|_1 + \tfrac{1}{2\lambda}\|x\|_2^2 \ \text{ such that } \ Ax = b$$

  if every i appears infinitely often

Sparse Kaczmarz and linearized Bregman

$$z^{k+1} = z^k - \frac{a_{r(k)}^T x_k - b_{r(k)}}{\|a_{r(k)}\|_2^2} a_{r(k)}$$
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- Two interesting things:
  1. Very similar to Kaczmarz. Other “minimum-J-solutions” possible?
  2. Very similar to linearized Bregman iteration.

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- Approach taken here: "Split feasibility problems" will answer the first and explain the second point.

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- **Split feasibility problems**
- Convergence rates

Convex split feasibility problems

- Split feasibility problem (SFP): Find x, such that

$$x \in \bigcap_{i=1}^{N_C} C_i, \qquad A_i x \in Q_i,\ \ i = 1, \dots, N_Q$$

C_i, Q_i convex sets, A_i linear

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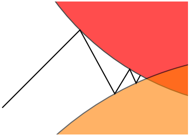

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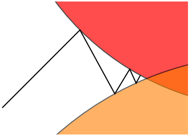

  $x^{k+1} = P_{C_i}(x^k)$
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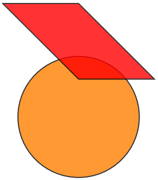
- [1933 von Neumann (two subspaces), 1962 Halperin (several subspaces), Dijkstra, Censor, Combettes, Bauschke, Borwein, Deutsch, Lewis, Luke...]

Tackling split feasibility problems

- Projecting onto $\{x \mid Ax \in Q\}$: Project onto separating hyperplane

$$H^k = \{x \mid \langle Ax^k - P_Q(Ax^k), Ax - P_Q(Ax^k)\rangle \leq 0\}$$

(separates x^k from $\{x \mid Ax \in Q\}$)



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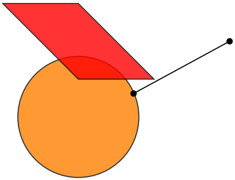

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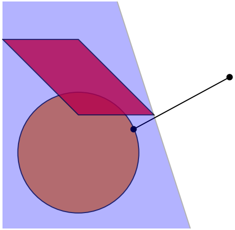

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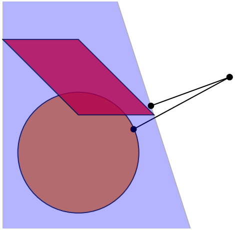

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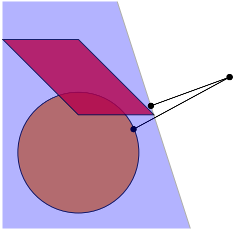

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  - $x^{k+1} = P_{C_i}(x^k)$
    for a constraint $u \in C_i$
  - $x^{k+1} = P_{H^k}(x^k)$
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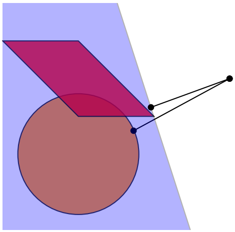
- Converges to feasible point.



Tackling split feasibility problems

- Projecting onto $\{x \mid Ax \in Q\}$: Project onto separating hyperplane

$$H^k = \{x \mid \langle Ax^k - P_Q(Ax^k), Ax - P_Q(Ax^k)\rangle \le 0\}$$

  (separates x^k from $\{x \mid Ax \in Q\}$)

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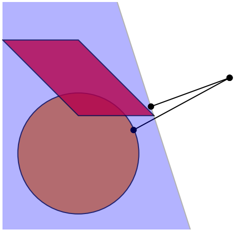
- Converges to feasible point.
- E.g.: $Q = \{b\}$: $x^{k+1} = x^k + t_k A^T(Ax^k - b)$
  $\rightsquigarrow$ minimum norm solution of $Ax = b$

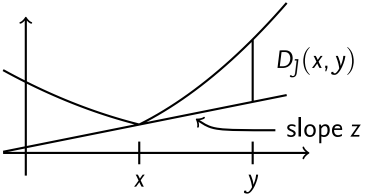


Towards sparse solutions with Bregman projections

- $P_C(x) = \operatorname{argmin}_{y \in C} \|x - y\|^2 \rightsquigarrow$ orthogonal projection
- $J : X \to \mathbf{R}$ convex, $z \in \partial J(x)$

$$D^z(x, y) = J(y) - J(x) - \langle z, y - x \rangle$$

Bregman distance



- Bregman distances $\rightsquigarrow$ *Bregman projection*:
$\Pi_C^z(x) = \operatorname{argmin}_{y \in C} D_J^z(x, y)$

Bregman projections

- Assume $J : \mathbf{R}^n \to \mathbf{R}$ continuous, α-strongly convex ($\implies \nabla J^*$ is α^{-1}-Lipschitz)
- Bregman projections onto hyperplanes $H = \{a^T x = \beta\}$ are simple: if $z \in \partial J(x)$

$$\Pi_H^z(x) = \nabla J^*(z - \bar{t}a), \quad \bar{t} = \operatorname*{argmin}_t J^*(z - ta) + t\beta$$

  Moreover: $z - \bar{t}a \in \partial J(\Pi_H^z(x))$ new subgradient in $\Pi_H^z(x)$.
- **RBPSFP**: Random Bregman projections for SFP $x \in \cap C_i$, $A_i x \in Q_i$:
  - Initialize $z_0 \in \partial J(x_0)$
  - $x^{k+1} = \Pi_{C_i}^{z^k}(x^k)$ or $x^{k+1} = \Pi_{H^i}^{z^k}(x^k)$, update $z^k \in \partial J(x^k)$
- random: every index appears infinitely often

Convergence

- **Theorem:** [Schöpfer, L., Wenger 2014] RBPSFP converges to a feasible point $\bar{x} \in C := \bigcap C_i \cap \{x \mid A_i x \in Q_i\}$.

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$$\min J(x) \text{ s.t. } Ax = b$$
Multiple possibilities, e.g.
  1. only one “difficult constraint”: $Ax \in Q = \{b\}$
  2. many simple constraints $C_i = \{a_i^T x = b_i\}$

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- In both cases: Convergence to minimum-J solution

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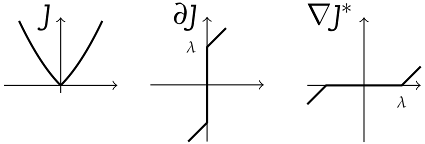
- $\nabla J^*(x) = (\partial J)^{-1}(x) = S_\lambda(x)$:



Basic algorithm and special cases:

- Variant 1: One difficult constraint $Ax = b$
- Variant 2: Many simple constraints $a_r^T x = b_r$
- In general: Block-processing $A_r x = b_r$

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Iteration:

- Calculate

$$
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z^{k+1} &= z^k - t_k A_r^T w^k \\
x^{k+1} &= \nabla J^*(z^{k+1})
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with appropriate stepsize t_k (depending on w^k and β_k)

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with appropriate stepsize t_k (depending on w^k and β_k)

- $J(x) = \|x\|_2^2/2$, variant 1.: Landweber iteration
- $J(x) = \|x\|_2^2/2$, variant 2.: Kaczmarz method
- $J(x) = \lambda\|x\|_1 + \|x\|_2^2/2$, variant 1.: Linearized Bregman
- $J(x) = \lambda\|x\|_1 + \|x\|_2^2/2$, variant 2.: Sparse Kaczmarz

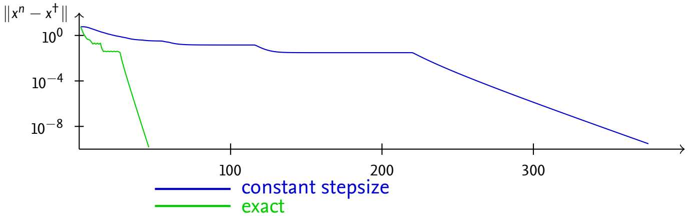
Inexact stepsizes are allowed

- Instead of projecting exactly, it suffices to move close enough
- Linearized Bregman:

$$t_k = \frac{\|Ax^k - b\|^2}{\|A^T(Ax^k - b)\|^2}, \quad \text{or} \quad t_k \leq \frac{1}{\|A\|^2}$$

- However: To compute exact stepsize, solve one-dimensional piecewise quadratic optimization problem (for $J(x) = \lambda\|x\|_1 + \|x\|_2^2/2$ can be done in $\mathcal{O}(n \log n)$, usually faster).

Stepsize comparison - linearized Bregman



$A \in \mathbf{R}^{1000\times 2000}$ Gaussian distriuted entries, $x^\dagger$ 20 non-zeros (also Gausian distributed)

- The Kaczmarz method
- Randomization
- Sparsity
- Split feasibility problems
- **Convergence rates**

Convergence rates for RBPSFP

Theorem (Schöpfer, L. 2018)

RBPSFB with $C = \bigcap C_i \cap \{x \mid A_i x \in Q_i\}$ *converges with a rate*

$$\mathbf{E}(\operatorname{dist}(x^k, C)) = \mathcal{O}(1/\sqrt{k})$$

if $\{C_k\}_k$ *and each* $\{Q_i, \operatorname{rg}(A_i)\}$ *is boundedly linearly regular and* J *is strongly convex.*
If, additionally, J *is piecewise linear quadratic, then method converges linearly, i.e.*

$$\mathbf{E}(\operatorname{dist}(x^k, C)) = \mathcal{O}(q^k).$$

Proof based on error bounds...
Corollary: The randomized sparse Kaczmarz (RaSK) method converges linearly.

Taylores results for randomized sparse Kaczmarz

Theorem (Schöpfer, L. 2018)

For RaSK with exact steps (ERaSK) for a consistent overdetermined system $Ax = b$ it holds that

$$\mathbf{E}(\|x^k - x^*\|_2) \le (1-\epsilon)^{k/2}\sqrt{2\lambda\|\hat{x}\|_1 + \|\hat{x}\|_2^2}$$

with

$$\epsilon = \frac{\tilde{\sigma}^2_{\min}(A)}{2\|A\|_F^2}\frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda}$$

where $\tilde{\sigma}_{\min} = \min\{\sigma_{\min}(A_J) \mid A_J \neq 0 \textit{ submatrix}\}$,
$|\hat{x}|_{\min} = \min\{|\hat{x}_j| \mid \hat{x}_j \neq 0\}$.

Randomized sparse Kaczmarz for noisy data

Following [Needell 2010] and [Lai, Yin 2013]:

Theorem

For $Ax = b^\delta$ with $\|b^\delta - b\|_2 \leq \delta$ it holds for RaSK

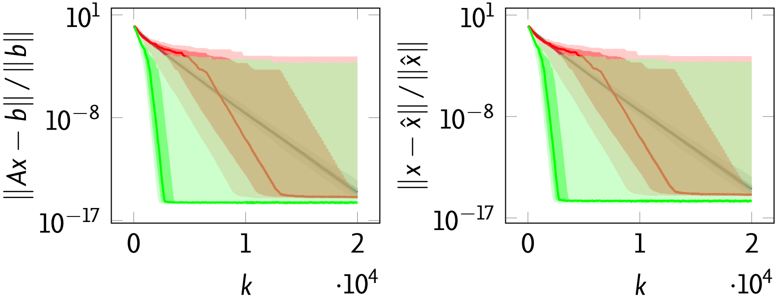
$$\mathbf{E}(\|x^k - x^*\|_2 \leq (1-\epsilon)^{k/2}\sqrt{2\lambda\|\hat{x}\|_1 + \|\hat{x}\|_2^2} + \sqrt{\frac{2|\hat{x}|_{\min} + 4\lambda}{|\hat{x}|_{\min}}}\frac{\delta}{\tilde{\sigma}_{\min}(A)}$$

and for ERaSK the upper bound is

$$(1-\epsilon)^{k/2}\sqrt{2\lambda\|\hat{x}\|_1 + \|\hat{x}\|_2^2} + \sqrt{\frac{2|\hat{x}|_{\min} + 4\lambda}{|\hat{x}|_{\min}}}\frac{\delta}{\tilde{\sigma}_{\min}(A)}\sqrt{1 + \frac{4\|A\|_{2,1}}{\delta}}$$

Sparsity also helps for overdetermined systems

- 200 columns, 1000 rows, **consistent** system $Ax = b$, unique solution $x^\dagger$, $\mathrm{nnz}(x^\dagger) = 25$

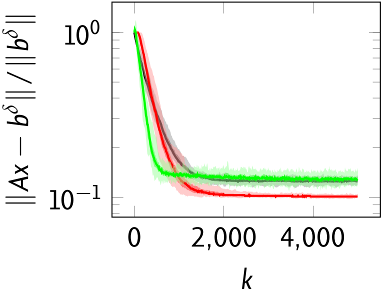


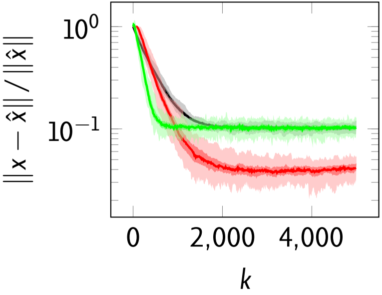
Black: Randomized Kaczmarz, Red: Randomized sparse Kaczmarz, Green: Exact-step randomized sparse Kaczmarz

Sparsity also helps for overdetermined systems

- 200 columns, 1000 rows, **inconsistent** system $Ax = b$,

10% relative error

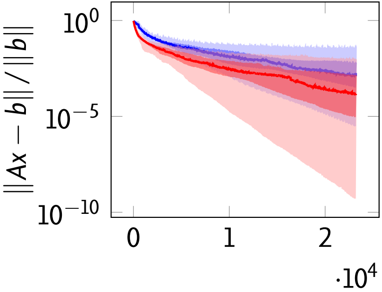


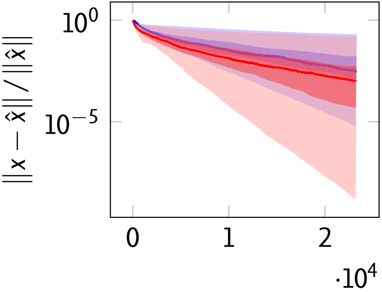


Black: Randomized Kaczmarz, Red: Randomized sparse Kaczmarz,
Green: Exact-step randomized sparse Kaczmarz

Randomization also helps

- Matrix from fan-beam CT, consistent system $Ax = b$, unique solution $x^\dagger$, 100 columns, 1164 rows, $\mathrm{nnz}(x^\dagger) = 20$

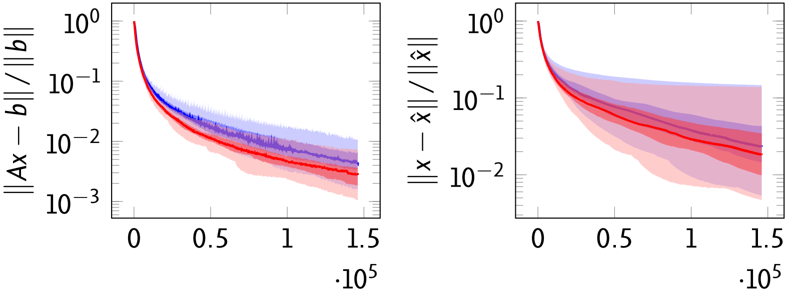




Blue: Sparse Kaczmarz, Red: Randomized sparse Kaczmarz

Randomization also helps

- Matrix from fan-beam CT, consistent system $Ax = b$, unique solution $x^\dagger$, 900 columns, 3660 rows, $\mathrm{nnz}(x^\dagger) = 180$



Blue: Sparse Kaczmarz, Red: Randomized sparse Kaczmarz

Conclusion

- Randomization gives uniform expected progress,
  hence convergence rates
- Randomization usually improves (random reshuffle also works)
- Extension to sparse solutions simple; exact stepsize matters, though
- Convergence of RaSK and ERaSK linear
- Exacts steps faster, lower accuracy for noisy data

References

- Dirk A. Lorenz, Frank Schöpfer, and Stephan Wenger, *The linearized Bregman method via split feasibility problems: Analysis and generalizations*, SIAM Journal on Imaging Sciences **2** (2014), no. 7, 1237–1262, [doi, arXiv].
- Dirk A Lorenz, Stephan Wenger, Frank Schöpfer, and Marcus Magnor, *A sparse Kaczmarz solver and a linearized Bregman method for online compressed sensing*, Image Processing (ICIP), 2014 IEEE International Conference on, IEEE, 2014, [doi, arXiv], pp. 1347–1351.
- Frank Schöpfer and Dirk A. Lorenz, *Linear convergence of the Randomized Sparse Kaczmarz method*, To appear in *Mathematical Programming*, 2018, [doi, arXiv].